Dynamics of Sliding Charge-Density Waves in \( 4 - \epsilon \) Dimensions

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The critical behavior of sliding charge-density waves just above threshold is analyzed. In \( 4 - \epsilon \) dimensions, the exponent \( \zeta \) defined through \( I_{\text{CDW}} = (V - V_T)^\zeta \), is found to be \( 1 - \epsilon/6 + O(\epsilon^2) \). Two different correlation lengths are found, with exponents \( \nu = 1/2 \) exactly and \( \nu_{\text{RS}} = 2/d + O(\epsilon^2) \). These results compare favorably with recent numerical results in two and three dimensions.

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The dynamics of charge-density waves (CDWs) pinned by impurities and pulled by an electric field [1] is one of the simplest and best-studied examples of a class of nonlinear collective transport phenomena involving motion of elastic media through random systems [2]. With a small applied force (proportional to the electric field), the CDW is pinned in one of a host of metastable configurations [1]. Above a sharp threshold force \( F_T \), however, the CDW slides with a nonzero mean velocity \( v \) giving rise to a nonlinear current proportional to \( v \) [1]. In the absence of thermal fluctuations and phase slip processes, both of which we will ignore here, the threshold and the steady-state velocity are nonhysteretic: Indeed, for the models we consider, there is a *unique periodic steady state* for any \( F > F_T \) [3]. The dynamic critical behavior for \( F \) just above \( F_T \) is the subject of this paper.

We are interested, in particular, in the exponent \( \zeta \) for the velocity \( v \sim f^\zeta \), where we have defined the reduced force \( f = F/F_T \). Extensive recent numerical simulations on a variety of models have yielded evidence for remarkable universality in two and three dimensions with \( \zeta(d=2) \approx 0.64 \pm 0.03 \) and \( \zeta(d=3) = 0.82 \pm 0.03 \) [4-6]. By contrast, mean-field theory yields nonuniversal behavior with \( 1 \leq \zeta_{\text{MF}} \leq \frac{1}{2} \) depending on the nature of the random pinning potential [4,7].

The origin of this nonuniversality is related to a crucial feature of these systems: Just above threshold the local motion is very jerky, consisting of slow motion superimposed on a series of "jumps." These jumps occur when slowly evolving local minima of the energy in which some segment of the CDW is moving disappear, causing the segment to quickly move forward to a new local minimum. Thus near threshold there are at least two important time scales: the period \( 2 \pi/\nu \) in which the phase of the CDW advances by \( 2\pi \) (i.e., the CDW moves by one wavelength) and the time scale, of order 1, of the fast jumps. To study the critical dynamics analytically, these time scales must both be handled.

In this paper, we perform a renormalization-group (RG) expansion around the "generic" mean-field dynamics with \( \zeta_{\text{MF}} = 1 \) and analyze the critical behavior to lowest order in \( d = 4 - \epsilon \) dimensions finding \( \zeta = 1 - \frac{1}{2} \epsilon + O(\epsilon^2) \) as well as two distinct correlation length exponents, and new scaling laws for correlation and response functions. Details will be presented in a longer paper [8].

We consider a class of models for the dynamics of the phases \( \varphi_x \) of the CDW at impurity sites \( x \). The dissipative equations of motion are [9]

\[
\frac{d\varphi_x}{dt} = \sum_y J_{xy}(\varphi_y - \varphi_x) + h_x Y(\varphi_x - \beta_x) + F, \tag{1}
\]

where the \( J_{xy} \geq 0 \) with \( \sum_x J_{xy} = 1 \) represent the elastic interactions. The pinning force is \( h_x Y(\varphi_x - \beta_x) = -\partial V_x(\varphi_x)/\partial \varphi_x \) with \( V_x \) the \( 2\pi \)-periodic local pinning potential with strength \( h_x \geq 0; \beta_x \), which is uniformly distributed on \([0,2\pi)\), is the preferred phase.

In mean-field theory, valid in the limit of infinite range \( J_{xy} \), each phase feels a smoothly varying elastic force from the mean field \( \bar{\varphi}(t) = \nu t \). For smooth \( V_x(\varphi) \), \( \zeta_{\text{MF}} = \frac{1}{2} \), while for scalloped potentials with linear-cusped maxima, e.g.,

\[
V_x(\varphi) = \frac{1}{2} h_x(\varphi_x - \beta_x)^2 \quad \text{for} \quad -\pi < \varphi_x - \beta_x < \pi, \tag{2}
\]

\( \zeta_{\text{MF}} = 1 \). This difference arises because, for smooth potentials in mean field, when the local minimum that a phase is in disappears, it has a long acceleration time before it jumps forward [7]. By contrast, there is no acceleration time for scalloped potentials [4]. In low dimensions, with short-range couplings, the elastic force on a phase near threshold is jerky rather than smoothly changing; it is dominated by jumps of a relatively small number of nearby phases. We thus expect that the slow response to slowly disappearing minima which occurs for smooth \( V_x \) in mean-field theory will not be relevant in low dimensions: The behavior should thus be more like that of the jump-dominated case of scalloped potentials. We thus expand about the mean-field behavior with scalloped potentials and \( \zeta_{\text{MF}} = 1 \). (The existence of a stable renormalization-group fixed point in \( 4 - \epsilon \) dimensions "near" this mean-field system supports our conjecture.)

To carry out an expansion about mean-field theory, we first introduce a generating function \( Z(l,\tilde{l}) \) for the dynamics involving \( \varphi(x,t) \) and a response field \( \tilde{\varphi}(x,t) \) [10]. Derivatives with respect to the conjugate fields \( l \) and \( \tilde{l} \) (at \( l = \tilde{l} = 0 \)) give the correlation functions of the phase \( \varphi \), and responses to perturbing forces added to Eq. (1). Because of causality, \( Z(l = 0,\tilde{l} = 0) = 1 \) [10], and hence averaged correlation and response functions are the derivatives of \( Z(l,\tilde{l}) \), the average of \( Z \) over the quenched
random variables $\beta$ and $h$. We next introduce fields $\Phi$ and $\hat{\Phi}$ which are essentially coarse-grained versions of $\varphi$ and $\hat{\varphi}$. Following Sompolinsky and Zippelius [11], we express $\tilde{Z}(0,0)$ as an expansion around the saddle point of $\Phi$ and $\hat{\Phi}$, which yields an expansion around mean field. The resulting effective action in $\tilde{Z} = \int D\Phi \int D\hat{\Phi} \exp(\tilde{S})$ is

$$\tilde{S} = \sum_{n=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! n!} \int dt_1 \cdots dt_{m+n} u_{m,n}(t_1, \ldots, t_m; t_{m+1}, \ldots, t_{m+n}) \hat{\Phi}_x(t_1) \cdots \hat{\Phi}_x(t_m) \Phi_x(t_{m+1}) \cdots \Phi_x(t_{m+n})$$

$$- \sum_{x,y} \int dt [J^{-1}]_{x,y} \hat{\Phi}_x(t) \Phi_y(t),$$

(3)

where $\Phi$ has been shifted by $\epsilon t$, so that $(\Phi) = 0$, and the vertices $u_{m,n}$ are obtained from the local response and correlations in mean-field theory:

$$u_{m,n} = \frac{\partial}{\partial \epsilon(t_{m+1})} \cdots \frac{\partial}{\partial \epsilon(t_{m+n})} \left( \varphi(t_1) \cdots \varphi(t_m) \right)_{\epsilon=0}.$$  

(4)

where $\varphi(t_1, \epsilon(t')), h, \beta$ is the solution to the local mean-field equation

$$\frac{d\varphi}{dt} = \varphi - \varphi + F + h Y(\varphi - \beta) + \epsilon(t)$$

(5)

with an extra perturbing force $\epsilon(t)$; the brackets denote averaging solutions to Eq. (5) over $\beta$ and $h$ and $c$ the truncated correlation and response functions. The vertex $u_{m,n}$ generally depends on $m+n-1$ time differences.

In Fourier space, the propagator, which arises from the second term of Eq. (3) and the $u_{1,1}$ vertex, becomes

$$- \int \int d^d q d\omega \hat{\Phi}(-q, -\omega) \Phi(q, \omega)$$

$$\times \{ -i\omega + q^2 + O(\omega^3, q^4) \},$$

$$U_{1,1} = \frac{1}{2\pi n!} \int \int \int d^d x dt_1 dt_2 \hat{\Phi}(x, t_1) \hat{\Phi}(x, t_2) \Phi(x, t_1) - \Phi(x, t_2)\}^n C^{(n)}(\epsilon t_1 - \epsilon t_2),$$

(7)

where $C^{(n)}(\epsilon t)$ is the $n$th derivative of $C(\epsilon t)$, which is the local (2$\pi$-periodic) mean-field correlation function

$$C(\epsilon t) \equiv \langle [\varphi(t) - \varphi] [\varphi(t + \epsilon) - \varphi(t + \epsilon)] \rangle_{\epsilon,t}.$$  

(8)

The $u_{m,n}$ vertices for $m > 2$ have similar low-frequency forms to Eq. (7). Note, however, that these forms have some $\Phi$ and $\hat{\Phi}$ at the same times; thus we must be very careful with the implications of causality. In addition, the jumps produce singularities in $C(\epsilon t)$ at short times, causing additional difficulties and essential physics.

We now integrate out the degrees of freedom in a momentum shell near the cutoff and rescale $x \rightarrow bx'$, $t \rightarrow b^2 t'$, $\hat{\Phi} \rightarrow \Phi^{\theta - d} \hat{\Phi}'$, and $\Phi \rightarrow b^\epsilon \Phi'$. With $\theta = 2$ and $\kappa = -\theta - d/2$, chosen to fix the propagator and $u_{2,2}$, we see that the whole $u_{2,m}$ series becomes marginal at $d = 4$, while the other operators remain irrelevant. To perform an expansion in $4 - \epsilon$ dimensions, we can drop these irrelevant operators after checking that at lowest order they do not feed back in a dangerous manner [8].

Coupled RG equations for the full set of low-frequency vertices $u_{2,m}$ can now be derived. Fortunately, these can be combined into a simple RG equation for the function $C(\varphi)$ which determines all these vertices via Eq. (7). This forces, however, the exponent $\kappa$ to be exactly zero for $d < 4$: This means that fluctuations in $\varphi$ scale in the same way as $\varphi$, i.e., the motion is jerky. We obtain, to one-loop order,

$$\frac{dC(\varphi)}{dt} = \frac{\epsilon + 2\theta + 2(z - 2)}{8\pi^2} \left\{ \frac{\partial C(\varphi)}{\partial \varphi} \right\}^2 + \frac{C(\varphi) - C(0)}{\partial \varphi^2}$$

(9)

Here, we have used the fact that, at the cutoff $\Lambda = 1$, the propagator $(q^2 - i\omega)^{-1}$ has a low-frequency form of $\delta(t)$. 3616
The physical fixed point function has the form
\[ C(\varphi) = \frac{1}{2} w[(\varphi - \pi)^2 - \pi^2/3] \] (10)
for \(0 < \varphi < 2\pi\), which has a linear cusp at \(\varphi = 2\pi m\). The fixed point value of \(w\) is \(w^* = 8\pi^2(2\pi + 20 + 2(z - 2))/3\). This form of \(C(\varphi)\) is exactly of the form found in mean-field theory in the limit \(v \to 0\) with scolaped parabolic pinning potentials as in Eq. (2) with \(w = [(h/(h+1))]^2\). The discontinuity in \(\partial C/\partial \varphi\) at \(\varphi = 0\) is a consequence of the jumps in \(\varphi\), which become sharp on the scale of the period \(2\pi/v\) as \(v \to 0\). Up to irrelevant operators, the system with this pinning potential and \(w = w^*\) is at the fixed point and can be used to calculate properties which need the full high-frequency structure of the dynamics.

In order to fix the exponents \(\theta\) and \(z\), we need to analyze the renormalization of the propagator. Because of the vanishing of \(U_{22}\), Eq. (8), as \(t_1 \to t_2\), the \(q^2\) part of the propagator has no nontrivial renormalization; as we will see this is true generally. This implies that \(z = 2 + \theta = 0\) and hence \(w^* = 8\pi^2/3\). The \(\Phi \partial \Phi/\partial t\) part of the propagator does, however, get renormalized by the \(u_{22}\) vertex. The contraction of one of the \(\Phi\)'s with one of the \(\Phi\)'s forces \(t_1 \to t_2\) to be of order 1 (i.e., \(\Lambda^{-2}\)) since \(\langle \Phi(t_1)\Phi(t_1) \rangle\) vanishes by the causality of the propagator. The adjustment of \(\theta\) to cancel the resulting renormalization yields
\[ \theta = 1/8\pi^2 \int dt_2 C^{(2)}(-i\epsilon t_2) t_2 e^{-i\epsilon t}. \] (11)

Unfortunately, the low-frequency form of \(C^{(2)}(-\epsilon t)\) is a \(\delta\) function at the origin, \(\sim \epsilon^{-1}\delta(\epsilon t)\) which should really spread over a time interval of order 1 from the full time dependence of \(u_{22}\). The integral in Eq. (11) is thus badly defined and appears to depend on the high-frequency behavior. Nevertheless, we can attempt to use the exact local mean-field correlation and response functions from the scolaped parabolic potential to analyze the behavior perturbatively in \(w\). The singular responses and correlations due to the local jumps are what give rise to the dangerous parts of \(u_{22}\): In particular, a perturbation applied just before a jump yields a large response after the jump because of the shift in the time of the jump. There are, however, no singular precursors to the jumps for scolaped potentials. Detailed calculations then show that causality combined with the above observation results in the \(\delta\)-function part of \(C^{(2)}\) being completely removed from the range of integration in Eq. (11). We can hence replace \(C^{(2)}(\epsilon t)\) by \(C^{(2)}(0^-) = w\) yielding
\[ \theta = w^* = \frac{\epsilon}{3} \quad \text{and} \quad z = 2 - \frac{\epsilon}{3}. \] (12)

We now have all the exponents at the fixed point, but still need to determine how deviations from the fixed point grow in order to determine \(\xi\) and the correlation length exponent \(\nu\). It is most convenient to work at fixed \(v\), with an extra counterterm \(\int d^d x dt [F-F(v)_{\text{MF}}] \Phi(x,t)\)
in the effective action which must be adjusted to make \(\langle \Phi \rangle = 0\), thereby determining \(F\). Fluctuations will change \(F_T\), but since they have no explicit dependence on \(F\), will not change the scaling of \(F - F_T\). Thus, from the scaling of \(\Phi\), we obtain
\[ \frac{d}{dt} \left[ F - F(v)_{\text{MF}} \right] = (z + \theta) \left[ F - F(v)_{\text{MF}} \right] + \text{const}, \] (13)
implying that
\[ v = \frac{1}{z + \theta} = \frac{1}{2}, \] (14)
which is in fact exact. Since the velocity scales like a frequency, we have
\[ \zeta = \frac{z}{z + \theta} = 1 - \frac{\epsilon}{3} + O(\epsilon^2). \] (15)

Scaling forms for correlation and response functions also follow, for example,
\[ \chi(q, \omega) \equiv \text{FT} \left( \frac{\partial \Phi(x,t)}{\partial x(t')} \right) \sim f^{-1} \chi(\omega \xi^z, q \xi). \] (16)
(The relation \(\zeta = z/z + \theta\) can also be derived using the \(q = 0\) limit of \(\chi\).) For \(q \xi \ll 1\) and \(\omega \xi \ll 1,
\[ \chi(q, \omega) \sim \frac{d\epsilon/dF}{-i\omega + Dq^2}, \] (17)
so that the diffusion coefficient \(D\) diverges as \(f^{-2}\) with \(f^{-1} \sim 1/q^2\). The statistical invariance of the equations of motion under a change of variables \(\varphi(x,t) \to \varphi(x,t) + ax\) implies \(\chi(q, \omega = 0) = 1/q^2\). This implies that \(D\) diverges in the same way as \(d\epsilon/dF\), yielding \(v = \frac{1}{2}\) exactly. The other limits of the scaling function \(X\) are rather subtle and we will not discuss them here [8].

The spatial correlations of Fourier components \(\Omega_m(x)\) of the periodic local velocities \(\partial \Phi(x)/\partial t\) decay as
\[ \langle \Omega_m(x) \Omega_{-m}(y) \rangle \sim \exp \left[ -(x - y)^2 \sqrt{\frac{m}{\xi}} \right], \] (18)
for \(m \neq 0\), with \(\xi \sim f^{-\nu}\), the distance which a disturbance will spread in one period. In addition to the dynamic correlations, one expects a static deformation of the phases \(\delta \varphi(x)\) defined as the time average of \(\varphi(x,t) - v t\) due to the persistent effects of the random potential. Surprisingly, in 4 - \(\epsilon\) dimensions, the static deformations scale differently from the dynamic correlations. This is caused by the growth under renormalization of a \(\varphi\)-independent part \(C_0\) of the function \(C(\varphi)\), which from Eq. (9) scales as \(b^{4-d-\eta}\) with \(\eta = 0 + O(\epsilon^2).\) The bare \(C_0\) is given by \(C_0 = \pi^2 [(h^2/(h+1))]^2 - (h^4/(h+1))^2\) for the scolaped parabolic potential. Because renormalizations of the \(\varphi\)-dependent parts of \(C(\varphi)\) always involve derivatives of \(C\) or \(\partial C(\varphi)/\partial \varphi(0)\), \(C_0\) does not feed back into the rest of \(C(\varphi)\). For \(\xi \gg 1\), the mean square static deformations are given by
\[ \langle [\delta \varphi(x) - \delta \varphi(x')]^2 \rangle \sim |x - x'|^{4-d-\eta}. \] (19)
The static local correlation $C_s$ also determines the scaling of the variance of threshold fields in systems of size $L$. We can define a finite-size scaling exponent $\nu_{FS}$ via $\langle (\delta F_L^2) \rangle^2 \sim L^{-2/\nu_{FS}}$. We then obtain

$$\frac{2}{\nu_{FS}} = d + \eta_s = d + O(\epsilon^2),$$

which therefore, to $O(\epsilon)$, saturates the rigorous inequality $\nu_{FS} \geq 2/d$ for probabilistic finite-size scaling lengths [12] in contrast to $\nu \leq 2/d$.

In order to perform calculations to higher order in $\epsilon$, ways must be found to systematize the treatment of both the jumps and slow time scales; some of the difficulties and possible outcomes will be analyzed in a longer paper [8]. It appears plausible, however, that the lowest-order results $\eta_s = 0$ and $\nu_{FS} = 2/d$ may hold to all orders in perturbation theory, albeit perhaps with nonperturbative corrections.

We now turn to comparison with other results. Recent numerical simulations of a variety of models in two and three dimensions have shown good evidence for universality [5,6], although for some of the models the critical regions appear to be rather small, and the apparent value of $\zeta$ is larger than one in $d = 3$ for quite wide ranges of $f$ [13]. Nevertheless, the best attempts to extract asymptotic results agree rather well with our $4 - \epsilon$ expansion result for $\zeta$ truncated at $O(\epsilon)$: $\zeta(d = 3) \approx \frac{5}{3}$ and $\zeta(d = 2) \approx \frac{1}{3}$. Preliminary measurements of the velocity-velocity correlations lead to estimates for $\nu(d = 2) \approx \nu(d = 3) \approx 0.5 \pm 0.1$ [6] in agreement with the exact result $\nu = \frac{1}{2}$. Note that in two dimensions $\nu$ seems to be well below $2/d$. Nevertheless, a finite-size scaling exponent extracted from the distribution of threshold fields yields $\nu_{FS}(d = 2) = 1.01 \pm 0.03$ and $\nu_{FS}(d = 1) = 2.01 \pm 0.02$ [4], consistent with $2/d$.

At this point, experimental data on sliding CDWs have been found to be fitted with $\zeta$'s in the range 1–1.5 down to 5% above $F_T$ [1,14]; a similar $\zeta_{\text{eff}}$ was also found in numerical simulations fitted over a comparable range [13]. The reason for the disagreement between experiments and the analytical results and more extensive numerical studies may thus simply be due to a narrow critical region (perhaps caused by the crossover from smooth potential to jump dominated behavior). Other effects that we have ignored, particularly phase slip processes [15] and thermal depinning, may, however, be important in the real experimental systems. We leave these questions, as well as those involving possible relationships to the critical behavior of CDWs below threshold and the dynamics of "sandpile" models, for future study [16].

In this paper, however, we have succeeded in analyzing the nonlinear critical dynamics of an extensive deterministic system—which is trivial asymptotically for any finite-size system—by a combination of field theoretical renormalization-group techniques and a boundary layer analysis of the effects of jumps, thus dealing with both the fast and the slow processes essential to the physics of this and many other extended nonequilibrium systems. We hope that the methods used—described in detail elsewhere [8]—will be fruitful for other applications.

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