Spin precession and energy conservation

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(Received 27 April 2000; published 5 September 2000)

In this paper, we examine the precession of a spin in a magnetic field, treating the particle and the applied magnetic field as a single closed system whose energy is conserved. Through two simple examples, it is shown that the standard expression for spin precession emerges only indirectly, and as a first-order approximate result. For the first example, deviations from the first-order result are easy to obtain, and are familiar from Stern-Gerlach spin separation. For the second example, deviations from the first-order result are not practically possible to obtain. Beyond these two examples, it is also shown within a semiclassical approach that the standard expression for spin precession in a magnetic field is always valid to first order, irrespective of how the Hamiltonian depends on other dynamical variables of the system.

PACS number(s): 03.65.-w

I. INTRODUCTION

The precession of a spin in an applied magnetic field is a simple application of quantum mechanics with wide-ranging practical applications. The precession is what one would expect from an analogy with classical mechanics and electromagnetism: the spin is associated with a magnetic moment \( \bm{\mu} \) that experiences a torque \( \bm{\mu} \times \bm{B} \) in a magnetic field \( \bm{B} \). Within quantum mechanics, spin precession is not obtained from a torque, but as a result of the energy difference between states with the spin parallel and antiparallel to the field. Thus if a spin \( \frac{1}{2} \) particle has a magnetic moment \( \bm{\mu} = \gamma \bm{S} \), the energy of the spin-up and spin-down states in a magnetic field \( \bm{B} \) are \( \pm \frac{\hbar}{2} \gamma \bm{B} \). A general state

\[
|\psi\rangle = c_1 |\uparrow\rangle + c_2 |\downarrow\rangle
\]

then evolves with time as

\[
|\psi(t)\rangle = c_1 e^{i \gamma B t / 2} |\uparrow\rangle + c_2 e^{-i \gamma B t / 2} |\downarrow\rangle.
\]

It is a standard calculation \[1\] to then verify that the projection of \( \langle \bm{S}(t) \rangle \) in the plane perpendicular to \( \bm{B} \) rotates in the plane with a frequency \( \omega = \gamma B \).

The crucial fact that this result relies on is that in an applied magnetic field the spin-up and spin-down states have different energies. This approach treats the magnetic field (and the source that produces it) as a “background.” On further thought, however, things do not seem so simple. It is easy to come up with examples where a particle does not experience a magnetic field in its initial and final states, but does at some intermediate stage in between. In such a situation, the energy of the particle in its initial state is independent of its spin orientation. Viewed as a single closed system, the particle with the magnetic field and the source of the magnetic field must have the same energy during the entire time evolution. The phase factor introduced by time evolution thus does not depend on the spin orientation of the particle. Since the spin-up and spin-down states do not pick up different phase factors under time evolution, it would seem that for a general spin state such as that in Eq. (1) there should be no time dependence to \( \langle \bm{S} \rangle \).

To illustrate the argument above, we consider two examples. In the first example, there is a long solenoid carrying a current. A spin \( \frac{1}{2} \) particle starts out as a wave packet outside the solenoid and traveling toward it. The final state is when the particle emerges on the far side of the solenoid. In the second example, a particle is inside a long solenoid. Initially, the solenoid has no current flowing through it. The current is then slowly increased to a maximum value, and then slowly lowered back to zero. In order to avoid any concerns that the explicit time dependence of the Hamiltonian might invalidate considerations based on energy conservation, the solenoid is connected to a capacitor. The resultant \( LC \) circuit is chosen to have a very small oscillation frequency, and the argument is applied to a half cycle of the solenoid current. In both these cases, the initial and final states clearly have an energy that is independent of the orientation of the spin of the particle.

In this paper, we analyze the occurrence of spin precession for these two systems, keeping track of conservation of total energy. We show that spin precession emerges indirectly in both cases, by mechanisms different from the standard analysis. Further, it is only a first-order effect in the strength of the magnetic field. For the first system, deviations from the first-order result are easy to generate, and can be understood quite simply in terms of Stern-Gerlach spin separation. For the second system, it is shown that it is essential to quantize the solenoid and capacitor variables to obtain even the first-order result. Deviations from this result, although possible in principle, are unattainable in practice. We also obtain a general semiclassical result, which includes these and other examples, to show how spin precession emerges as a first-order effect in the magnetic field.

II. TIME-INDEPENDENT FIELD

Consider a long solenoid carrying a fixed (time-independent) current, with a neutral spin \( \frac{1}{2} \) particle traveling directly toward it from outside. For definiteness, the axis of the solenoid lies along the \( z \) axis \( \{ \text{i.e., } (x = 0, y = 0) \} \), extending from \( z = -L \) to \( L \). The particle starts out at \( (-a,0,0) \), moving in the \( x \) direction. (See Fig. 1.) Let the spin of the particle initially point along \( x \). If \( L \gg a \), the magnetic field...
The spin-up particle slows down slightly inside the solenoid, and the spin-down particle speeds up. The energy of the rest of the system, i.e., the solenoid and the magnetic field, is conserved separately. There is a slight change in the energy stored in the magnetic field when the particle enters the solenoid, which is equal to the work done by the current source driving the solenoid against the Faraday electromotive force (emf). Thus the wave function associated with the particle can be isolated from that of the background in this case.

Both the $|\psi\rangle_\uparrow$ and $|\psi\rangle_\downarrow$ parts of the wave function can be evolved with time. In terms of their components, both $f_p$ and $g_p$ pick up the same phase $\exp[-ip^2t/(2m\hbar)]$. However, the overall evolution of $|\psi\rangle_\uparrow$ and $|\psi\rangle_\downarrow$ is slightly different, owing to the different spatial dependence of $f_p$ and $g_p$. To first order in the scattering potential, the spin-up part travels forward as a localized packet, moving slower when it is inside the solenoid. The same is true for the spin-down part, except that it travels faster inside the solenoid. On the far side of the solenoid, the $|\psi\rangle_\uparrow$ and $|\psi\rangle_\downarrow$ parts of the wave function are spatially displaced relative to each other. One can solve explicitly for $f_p(x)$ and $g_p(x)$ and verify that the phase of $f_p(x)$ at any $x$ on the far side of the solenoid lags behind that of $g_p(x)$. This leads to an effective phase difference between $|\psi\rangle_\uparrow$ and $|\psi\rangle_\downarrow$, because $\langle S \rangle$ (or any operator associated with the particle) involves the relative phase of $\langle x|\psi\rangle_\uparrow$ and $\langle x|\psi\rangle_\downarrow$ for the same $x$. If $w$ is the length of the solenoid and $\delta p$ is the momentum difference between up and down spin inside the solenoid, the phase difference is $\delta \phi = w \delta p / \hbar = (\delta E) \tau / \hbar$, where $\delta E$ is the difference in kinetic energy inside the solenoid and $\tau$ is the time of flight through it. Since the difference in kinetic energy $\delta E$ is equal to $\gamma h B$, one obtains the standard result.

It is easy to see that this result is only valid to first order in $B$. For very large magnetic fields, in addition to scattering from the edges of the solenoid, $\delta \phi$ is sufficiently large that the two parts of the wave packet separate completely, and $\langle S \rangle = 0$. For smaller magnetic fields, the partial separation between the spin-up and spin-down part of the wave packet leads to an attenuation of the interference between them. This is not surprising: the localized magnetic field due to the solenoid is just a special case of a spatially varying magnetic field. For any magnetic field whose magnitude varies spatially, there is Stern-Gerlach separation of the spin-up and spin-down parts of the particle wave function. For this system, we see that spin precession and the attenuation of interference are two aspects of the same phenomenon.

We note here that if the particle traveling through the solenoid is charged, it also experiences a Lorentz force that bends its trajectory and thereby limits how long it can spend inside the solenoid. A semiclassical calculation shows that the spatial separation between the up- and down-spin parts of the wave packet is $\delta x = (\delta \phi) \tau = \gamma h B \tau / p_0 < \gamma h B / (p_0 \omega_e)$, where $\omega_e$ is the cyclotron frequency. In particular, for electrons, $\delta x < \hbar / p_0 \approx \hbar / (\Delta p) \ll \Delta x$, where $\Delta p$ and $\Delta x$ are the momentum and position uncertainties of the wave packet. Thus free electrons cannot be significantly spin separated by passing them through a solenoid.
III. TIME-DEPENDENT FIELD

For the second example, we consider a particle inside a long solenoid oriented along the z axis. The particle is localized sufficiently well so that it experiences a spatially uniform magnetic field. However, the magnetic field changes with time. This is complementary to the previous example of a spatially varying, but time-independent magnetic field.

The time dependence of the magnetic field is achieved by coupling the solenoid (with inductance L) to a capacitor (with capacitance C), to form an LC circuit. The current in the solenoid is \( I(t) = I_0 \sin(\omega t) \), with \( \omega^2 = 1/(LC) \). At time \( t = 0 \), \( I(t) = 0 \) and the particle spin is oriented along x. As in the previous example, the energy of the particle is independent of the spin orientation at \( t = 0 \). Unlike the previous example, this is not true for general t. However, the change in the particle energy is compensated by a corresponding dependence on spin orientation of the energy stored in the magnetic field. Thus the total energy of the system remains independent of spin orientation. We wish to obtain the change in the orientation of the spin due to precession after a half cycle, i.e., at \( t = \pi/\omega \).

We start with the classical Lagrangian for a magnetic moment in an LC circuit, proceed to the Hamiltonian, and then consider the quantum problem. The Lagrangian for the system is

\[
\mathcal{L} = \frac{1}{2} LQ^2 - \frac{Q^2}{2C} + \lambda \gamma S \dot{Q} \cos \theta.
\]

Here \( Q \) is the charge on the capacitor, \( \dot{Q} \) is the current flowing in the solenoid, and \( \theta \) is the orientation of the spin with respect to the z axis. The magnetic field in the solenoid is \( \lambda \dot{Q} \) in the \( z \) direction \( (\lambda > 0) \), which defines \( \lambda \). The equation of motion for \( \dot{Q} \) from this Lagrangian is

\[
L \dot{Q} = -\dot{Q}/C + \lambda \gamma S \dot{Q} \sin \theta.
\]

The last term in this equation can be verified to be equal to the Faraday emf generated by the rate of change of \( \theta \).

The canonical momentum associated with the charge \( Q \) is

\[
P_Q = LQ + \lambda \gamma S \cos \theta
\]

from which the Hamiltonian is

\[
\mathcal{H} = \frac{1}{2L} (P_Q - \lambda \gamma S \cos \theta)^2 + \frac{1}{2C} Q^2.
\]

For the quantum version of this problem, we replace \( P_Q \) with \( q \) and \( Q \) with \( i \hbar \partial_q \). Also, \( S \cos \theta \) is replaced with \( S_z \). Thus the quantum Hamiltonian is

\[
H = -\frac{\hbar^2}{2C} \partial_q^2 + \frac{1}{2L} (q - \frac{i}{2} \epsilon \sigma_z)^2,
\]

where we have replaced \( \lambda \gamma \hbar \) with \( \epsilon \).

We see that \( L \dot{Q} \) is not the canonical momentum associated with the charge \( Q \). This has significant implications in quantum mechanics. Recall that interference between two parts of the wave function associated with the particle only occurs if the associated “background” wave functions (corresponding to the solenoid and capacitor) are the same for both parts. One might naively think that the background wave function should have the same value of \( Q \) for both interfering parts, but this is not the case. Physically, this is so because the total magnetic field \( \mathbf{B} \) is the fundamental dynamical variable, rather than the magnetic field due to the current flowing in the solenoid [3]. The ratio of the two terms in \( P_Q \) is simply the ratio of the \((z\) component of) the magnetic moment of the particle to that of the solenoid. A fully quantum treatment of this problem would not use the lumped variable \( P_Q \) to characterize the magnetic field, but instead treat \( \mathbf{B} \) as a quantum field.

As before, we consider the spin-up and spin-down states separately. From the time evolution of both of them, we can obtain the time evolution of any initial spin state. For definiteness, we consider the case when the spin initially points along \( x \), i.e., when \( \langle |\psi\rangle \) is initially equal to \( \{|\uparrow\rangle + |\downarrow\rangle\}/\sqrt{2} \). From Eq. (7), we see that the \( |\uparrow\rangle \) and \( |\downarrow\rangle \) cases both correspond to harmonic-oscillator Hamiltonians for the LC circuit, but with the equilibrium value of \( q \) at \( \pm \epsilon/2 \). This is shown in Fig. 2. Let the initial state of the circuit be a Gaussian wave packet centered at \( q = 0 \) and traveling to the right, after a half cycle of the oscillators, the spin-up and spin-down wave packets are centered at \( \epsilon \) and \(-\epsilon \) respectively, and are traveling to the left.

FIG. 2. Harmonic-oscillator wells corresponding to spin up and spin down, with the Hamiltonian of Eq. (7). If the initial state of the system is a wave packet centered at \( q = 0 \) and traveling to the right, after a half cycle of the oscillators, the spin-up and spin-down wave packets are centered at \( \epsilon \) and \(-\epsilon \) respectively, and are traveling to the left.
Eq. (2), the relative phase between $|\psi_\uparrow\rangle$ and $|\psi_\downarrow\rangle$ should then be $2\gamma\lambda p_0$, which is the same as $2\epsilon p_0/\hbar$. A similar result can be obtained for arbitrary $t$.

As in the first example we considered, the relative phase difference between $|\psi_\uparrow\rangle$ and $|\psi_\downarrow\rangle$ (and therefore spin precession) emerges indirectly, rather than from any explicit difference in their time-dependent prefactors. In order to obtain the correct result, it is essential to treat the solenoid current as a dynamical variable rather than a background parameter. As we see, $q = P\dot{Q}$ is the appropriate coordinate to work with, rather than $\dot{Q}$. If interference had occurred between parts of the wave function with the same value of $\dot{Q}$ rather than $q$, the splitting between the two potential wells in Fig. 2 would have been effectively eliminated.

The standard expression of Eq. (2) is obtained to first order in $\epsilon$. Once again, it is clear that this is only a first-order result. If $\epsilon$ were large, $|\psi_\uparrow\rangle$ and $|\psi_\downarrow\rangle$ would be sufficiently shifted in $q$ with respect to each other for there to be no overlap between them. Physically, this is a comparison between the uncertainty in the magnetic moment of the solenoid (due to an uncertainty in the current flowing through it) compared to the spin magnetic moment. In practice, it is quite unrealistic to try to determine the magnetic moment of the solenoid so accurately, even though it is not impossible in principle.

Interestingly, the result obtained in this section does not apply when the external magnetic field is due to a magnet rather than a solenoid. This is because, in addition to the field energy and the energy of the test particle, the energy of the magnet depends on the magnetic field due to the test particle, and thus its spin orientation. It is straightforward to verify that this term is equal to $-\gamma S \cdot B_0$, where $B_0$ is the field due to the magnet at the test particle, and $S$ is the spin of the test particle. This cancels the $-\gamma S \cdot B_0$ term in the field energy that was discussed at the beginning of this section. Thus the total energy of the entire system has the standard $-\gamma S \cdot B_0$ spin dependence.

**IV. GENERAL SEMICLASSICAL RESULT**

The examples considered in the previous two sections have the advantage that everything can be calculated explicitly. It is natural to ask whether the results obtained are more general: will spin precession be (effectively) seen for any possible system? In this section, we show that, within a semiclassical approximation, this is generally true to first order in the spin-dependent term in the Hamiltonian.

Consider a system with Hamiltonian $H_+(p,q)$, compared to one with a Hamiltonian $H_-(p,q)$. In the first example we considered, $p$ and $q$ would be the particle coordinates. The difference between the two Hamiltonians would be the difference in the spin-up and spin-down Hamiltonians. Since the magnetic field is nonzero only inside the solenoid, $\Delta H = H_+ - H_-$ would be a function of $q$, but it would be independent of $p$. In the second example, $\Delta H$ is once again only a function of $q$, and is equal to $\epsilon q L$.

Let both systems start out with the same expectation value for the energy, i.e., $\langle \Delta H \rangle = 0$. Let the initial wave packets for both systems be of the form $F(q - q_i)\exp[ip_i(q - q_i)/\hbar]$, where $F$ is a function with a maximum at $q = q_i$ and concentrated around this point. (The standard choice is for $F$ to be a Gaussian.) Thus the initial position of both systems is peaked at $q_i$, and the initial momentum is peaked at $p_i$. After a time $t$, the final wave functions of the two systems are then

$$\psi_\pm(q',t) \sim \int dq F(q - q_i)\exp[ip_i(q - q_i)/\hbar] \exp[iS_\pm(q',q_i,t)/\hbar],$$

where $S_\pm(q',q_i,t)$ are the classical actions for the systems to start from $q$ and end at $q'$ after a time $t$, and we have suppressed Jacobian prefactors to the integral. Since the integral over $q$ is peaked around $q_i$ and $-\partial_q S_\pm$ is equal to the initial momentum required to go from $q$ to $q'$ in a time $t$, we see that the final wave function is peaked around $q_f$, where $q_f$ are the final coordinates for the (classical) systems that have Hamiltonians $H_\pm$ and are initially at $(q_i,p_i)$. Also, since $\partial_q S_\pm$ are the final momenta associated with the two systems, the phase of the wave function varies with $q'$ as $\phi = [p_f - (q' - q_f)]/\hbar$ for the two systems. Thus the final wave functions of the two systems have the form

$$\psi_\pm(q',t) \sim \exp[i\phi]G(q' - q_f)$$

with

$$\phi = [p_f(q' - q_f) + S_\pm(q_f , q_i , t)]/\hbar,$$

where $G$ is a function peaked when its argument is zero.

We are interested in the phase difference between $\psi_+(q',t)$ and $\psi_-(q',t)$. For small $\Delta H$, we can compare the exponentials in Eq. (9) to first order in $\Delta H$. Defining $p_{f+} = p_f + \delta p_f$ and $q_{f+} = q_f + \delta q_f$, we find that the phase difference $\delta \phi$ between $\psi_+$ and $\psi_-$ is

$$\delta \phi(t) = \delta \phi_{q}(q' - q_f) - \delta \phi_{p} = \delta \phi_{q} + \delta \phi_{p} = \delta S/\hbar.$$

The first term on the right-hand side vanishes, since $q' = q_f$ and we are only calculating $\delta \phi$ to first order. To simplify the resulting expression further, it is easiest to take the time derivative, yielding

$$\hbar \frac{d \delta \phi(t)}{dt} = -\dot{p}_f \delta q_f - \dot{q}_f \delta p_f + \delta (p_f q_f - H).$$

Since energy is conserved, and is the same for both systems in the initial state, $\delta H = 0$. Eq. (11) thus yields

$$\hbar \frac{d \delta \phi(t)}{dt} = -\dot{p}_f \delta q_f + \dot{q}_f \delta p_f.$$

The right-hand side of this equation can be expressed as

$$\delta \phi = \delta \phi_{q} + \delta \phi_{p} = \delta \phi_{q} + \delta \phi_{p} = \delta S/\hbar.$$

Thus

$$\hbar \frac{d \delta \phi(t)}{dt} = H_+(p_{f+},q_{f+}) - H_+(p_{f-},q_{f-}).$$
Since $H_1(p_{f_+}, q_{f+}) = H_2(p_{f-}, q_{f-})$, this simplifies to
\[
\hbar \frac{d\delta f(t)}{dt} = -\Delta H(p_f, q_f),
\]
which is the standard result.

V. CONCLUSION

In this paper, we have reexamined the precession of a spin in a magnetic field. In the standard approach, this is done with the applied magnetic field treated as a background, in which the particle then has an energy that depends on its spin orientation. In this paper, we have instead considered the particle and the applied magnetic field, as well as the source of the magnetic field, as a single closed system whose energy is conserved. We have seen through examples that the standard expression for spin precession emerges only indirectly, not due to an energy difference between the up and down states. Further, the standard expression is only a first-order approximate result. Whether corrections to this result are observable depends on the specific system one is considering. We have also seen that for a spin in a magnetic field, regardless of the details of the total system being considered, the standard expression for spin precession is always valid to first order.

ACKNOWLEDGMENT

I thank Harsh Mathur for several useful discussions.

[2] In an actual classical system, there would be kinetic-energy terms associated with the magnetic moment too. However, there is no quantum analog for such kinetic terms for the spin magnetic moment, so we ignore them.
[3] Another way to see this is to note that if an extra field is turned on to pull the electron sideways, when it emerges from the solenoid the Faraday emf changes the current flowing in the solenoid by $\lambda y S_\epsilon$, which is different for the spin up and spin down case. Since with the electron outside the solenoid, interference should occur with the same $Q$ for the up spin and down spin parts of the wave function, $Q$ must be different for up and down spin parts when the electron is inside the solenoid.
[4] If the initial state is specified by $LQ = q = 0$, there would instead be different wave packets for $|\uparrow\rangle$ and $|\downarrow\rangle$, shifted slightly in $q$ with respect to each other. This would still yield the same first-order result for spin precession as obtained in this section.